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# Technical Report

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## Abstract

This paper presents the derivation new expressions for the statistics of a Chi-square distribution with $n$ degrees of freedom and where n is an even number. The complex Gaussian components of the chi-square distribution are modelled with a linear correlated model using different statistics (multi-rate) for each component. We focus on the specific expressions for the probability density function (PDF) and complementary cumulative density function (CCDF). Unlike previous approaches, we use a frequency domain interpretation that allows us to derive a closed form expression for the characteristic function (CF) as an inverse polynomial equation. Using the roots of this polynomial equation, it is possible to decompose the CF as a partial fraction expansion (PFE). This allows us to obtain a simple expression for both the PDF and CCDF by simply using the inverse Fourier transform of PFE decomposition of the CF. The statistics derived here have a much lower complexity than the expressions obtained from conventional non-frequency domain methods at the expense of the complexity of the polynomial root solution scheme. In scenarios where the average statistics of the components do not change over some periods of time, the proposed expressions provide the lowest possible complexity, as the polynomial rooting process needs to be conducted only once and potentially offline.

# New expressions for the central Chi-square distribution with even degrees of freedom and correlated multivariate complex components

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***Abstract*—This paper presents the derivation new expressions for the statistics of a Chi-square distribution with $n$ degrees of freedom and where $n$ is an even number. The complex Gaussian components of the chi-square distribution are modelled with a linear correlated model using different statistics (multi-rate) for each component. We focus on the specific expressions for the probability density function (PDF) and complementary cumulative density function (CCDF). Unlike previous approaches, we use a frequency domain interpretation that allows us to derive a closed form expression for the characteristic function (CF) as an inverse polynomial equation. Using the roots of this polynomial equation, it is possible to decompose the CF as a partial fraction expansion (PFE). This allows us to obtain a simple expression for both the PDF and CCDF by simply using the inverse Fourier transform of PFE decomposition of the CF. The statistics derived here have a much lower complexity than the expressions obtained from conventional non-frequency domain methods at the expense of the complexity of the polynomial root solution scheme. In scenarios where the average statistics of the components do not change over some periods of time, the proposed expressions provide the lowest possible complexity, as the polynomial rooting process needs to be conducted only once and potentially offline.**

## I. INTRODUCTION

The central chi square distribution has many different types of applications in science and engineering. In communications, the chi-square distribution is mainly used for modelling of fading channels and multiple antenna receivers(see [1]-[9]). The main statistics for this distribution are well known and have been studied for long time. However, under the assumption of correlation, the analysis gets complicated and in general not many works exist that address the full statistics analysis due to the complexity of calculation.

This paper provides a new method of addressing the problem using a frequency domain interpretation of the characteristic function of a chi square distribution with even degrees of freedom and correlated complex components. *Notation*: $f_{x|y}(x)$, $F_{x|y}(x)$ and $\bar{F}x|y(x)$ denote, respectively, the probability density, cumulative distribution, and complementary cumulative distribution function of the random variable x conditioned on the random variable $y$. $(\cdot)^H$ and $(\cdot)^T$ are the transpose and Hermitian vector transpose operators, respectively. $\omega$ is the frequency domain variable in radians, and $i = \sqrt{-1}$. $\Psi(i\omega)$ is the characteristic function of the random variable $x$.

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## II. SYSTEM MODEL

### A. Scenario and signal model

Consider the Gaussian complex circular variable $h_m$ with variance $\gamma_m$: $\mathcal{CN}(0, \gamma_m)$. For convenience, all variables will be expressed using a linear correlation model:

$$h_m = \sqrt{\gamma_m}(\sqrt{1-\lambda_m^2}Z_m + \lambda_m G), \tag{1}$$

where the variables $Z_m$, $G$ are identically and independently distributed (*i.i.d.*) zero-mean complex circular symmetrical Gaussian random variables with unit variance. The correlation coefficient can be defined as follows:

$$\rho_{m,\tilde{m}} = E[h_m^* h_{\tilde{m}}] = \frac{\lambda_m^* \lambda_{\tilde{m}}}{\gamma_m \gamma_{\tilde{m}}}.$$

## III. STATISTICS

Consider the summation of the squares of $M$ random circular Gaussian random variables:

$$\xi = \sum_{m=1}^{M} |h_m|^2$$

By substituting the correlation model of (1) in the previous expression we obtain:

$$\xi = \sum_{m=1}^{M} \gamma_m \left| \sqrt{1-\lambda_m^2} Z_m + \lambda_m G \right|^2 \tag{2}$$

Consider now the previous expression conditional on an instance of the random variable $G$. Under this assumption, the expression in (2) becomes the summation of the squares of Gaussian complex variables $\sqrt{\gamma_m(1-\lambda_m^2)}Z_m$ each one with a mean given by $\sqrt{\gamma_m}\lambda_m G$. Therefore, the variable $\xi$ conditional on an instance of random variables $G$ has a non-central chi-square distribution with $M$ degrees of freedom. The conditional characteristic function (CF) of $\xi$ can be thus written as (see [10] for details of chi-square distributions):

$$\Psi_{\xi|G}(i\omega) = \prod_{m=1}^{M} \frac{1}{(1-i\omega\tilde{\gamma}_m)} e^{\frac{i\omega|\tilde{\lambda}_m G!^2}{1-i\omega\tilde{\gamma}_m}},$$

where $\tilde{\gamma}_m = \gamma_m(1-\lambda_m^2)$ and $\tilde{\lambda}_m = \sqrt{\gamma}_m \lambda_m$. Averaging the previous expression over the PDF of the random variable

$G$ yields the expression for the unconditioned characteristic function:

$$\Psi_\xi(i\omega) = \frac{1}{\sum_{\tilde{n}} \sum_m |\lambda_m|^{-2} \prod_m (1 - i\omega\tilde{\gamma}_m)}.$$

Using partial fraction expansion, the previous expression becomes:

$$\Psi_\xi(i\omega) = \sum_{k=1}^{M} \frac{A_k}{1 - i\omega\gamma_k}. \tag{3}$$

where $A_k = \gamma_k \prod_{\tilde{k}\neq k} \left(1 - \tilde{\gamma}_{\tilde{k}}/\breve{\gamma}_{\tilde{k}}\right)$, and $\gamma_k$ is the $k$th root of the polynomial function of the denominator. The back-transform of (3) yields a complementary cumulative distribution function (CCDF) given by:

$$\bar{F}_{\hat{\Gamma}}(y) = \sum_{k=1}^{M} A_k e^{-\frac{y}{\gamma_k}}. \tag{4}$$

This concludes the derivation of the statistics of the chi-square distribution with correlated complex circular g¡aussian processes.

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